1984
New York State Mathematics League

POWER QUESTION

1. A sequence of positive integers $a_1, a_2, a_3, \ldots$ is defined as follows:

$$a_{n+1} = a_n - S_n, \text{ where } S_n \text{ is the sum of the digits of } a_n.$$

If $100 < a_1 < 1000$, show that the sequence must contain the number 63.

2. A sequence $a_1, a_2, a_3, \ldots$ is defined as follows:

a_1, a_2, and a_3 are positive integers;
a_n, a_{n+1}, and a_{n+2} form a geometric progression if n is odd (but the ratio must not be ½);
a_n, a_{n+1}, and a_{n+2} form an arithmetic progression if n is even.

(a) Will all the terms of such a sequence always be integers? Justify your answer.

(b) Show that $\frac{a_n}{a_1}$, for odd n, will always be the square of a rational number.

3. The sequence $\frac{n!}{1}, \frac{n!}{2}, \frac{n!}{3}, \ldots, \frac{n!}{k}, \ldots$ is to continue so long as $\frac{n!}{k}$ is an integer. Thus $\frac{14!}{1}, \frac{14!}{2}, \frac{14!}{3}, \ldots, \frac{14!}{16}$ is the full sequence for $n = 14$. Show that $n!$, for $n \geq 5$, is divisible by the positive integers 1, 2, 3,. . .,k where $k + 1$ is the smallest prime greater than n. (*Note:* One method of doing this problem requires the use of the theorem: "For any integer $x \geq 2$, there is always a prime between x and $2x$.")

4. A sequence of positive integers $a_1, a_2, a_3, \ldots$ is defined as follows:

$$a_{n+1} = a_n + P_n, \text{ where } P_n \text{ is the product of the digits of } a_n.$$

The sequence ends if ever $a_{n+1} = a_n$. Prove that, if $a_1 < 1\,000\,000$, the sequence must end. (If a_n consists of a single digit, then $P_n = a_n$.)