1990
American Regions Mathematics League

Power Question—Diophantine Equations

In the following equations, *all* letters represent positive integers, and $a > b$.

Let us examine the expression a^3+b^3, where $a > b$. One well-known result is that $a^3+b^3 = c^3$ has no solution in positive integers. For each of the equations in parts I and II, either:

1. Prove that no solutions can exist OR
2. Show how an infinite number of solutions can be generated.

I. A. $a^3 + b^3 = c^2$

B. $a^3 + b^3 = c \cdot d \cdot e$, where c, d, and e are in geometric progression

C. $a^3 + b^3 = c \cdot d \cdot e$, where c, d, and e are in arithmetic progression

D. $a^3 + b^3 = 3p$, where p is a prime greater than 3

II. A. $a^3 + b^3 = 2^c$

B. $a^3 + b^3 = 3^c$

C. $a^3 + b^3 = p^c$, where p is a prime greater than 3

III. Assuming that $a^3 + b^3 = c!$ has solutions, and c is at least 12:

A. Prove that the largest prime less than c does not divide a.

B. Prove that $a + b$ is a multiple of 330.