Power Question—Diophantine Equations

In the following equations, all letters represent positive integers, and $a > b$.

Let us examine the expression $a^3 + b^3$, where $a > b$. One well-known result is that $a^3 + b^3 = c^3$ has no solution in positive integers. For each of the equations in parts I and II, either:

1. Prove that no solutions can exist OR
2. Show how an infinite number of solutions can be generated.

I.  
A. $a^3 + b^3 = c^2$
B. $a^3 + b^3 = c \cdot d \cdot e$,
   where $c, d,$ and $e$ are in geometric progression
C. $a^3 + b^3 = c \cdot d \cdot e$,
   where $c, d,$ and $e$ are in arithmetic progression
D. $a^3 + b^3 = 3p$, where $p$ is a prime greater than 3

II. 
A. $a^3 + b^3 = 2^c$
B. $a^3 + b^3 = 3^c$
C. $a^3 + b^3 = p^c$, where $p$ is a prime greater than 3

III. Assuming that $a^3 + b^3 = c!$ has solutions, and $c$ is at least 12:
A. Prove that the largest prime less than $c$ does not divide $a$.
B. Prove that $a + b$ is a multiple of 330.